Generating Trigonometric Identities by Inductive Reasoning, Using Super Hexagon and Its Application

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Abstract

Trigonometric functions are basic mathematical objects that all students and academics at different level of education who are studying science and engineering try to tame. Usually, referring to them brings to mind difficulties of memorizing connections linking them. This paper discusses how a basic tool "super hexagon", a hexagonal diagram whose vertices are labeled using the six basic trigonometric ratios to generate some trigonometric identities by inductive reasoning. This will hasten the understanding of undergraduates in finding the derivatives of trigonometric functions. It also suggests an alternative way to show the relationships between the main trigonometric functions, in order to help in understanding them as well as in visualizing the various connections. Hence, apply them in mathematical models of their respective disciplines in applied sciences.

Keywords: *Deductive reasoning, Inductive reasoning, Trigonometry, Trigonometry Identities, Super Hexagon, Calculus, Differential Calculus*

1.0 Introduction

An effective way to rely less on memory for learning is through the use of visualization. Visualization means seeing an image of something in your mind. Different people do not always "see" things in the same way, but "visual thinking" is very important in deepening students' understanding (Dörfler, 1991).

1.1 Deductive Reasoning.

A deductive argument is one that the arguer intends to be (deductively) valid, that is, to show a guarantee of the validity of the conclusion if the premises (assumptions) of the argument are true. This notion may also be conveyed by stating that the premises of a deductive argument are designed to give such strong evidence for the conclusion to be erroneous. A (deductively) valid argument is one in which the premises succeed in ensuring the conclusion. A valid argument is considered to be sound if its premises are true.

1.2 Inductive Reasoning

The distinction between the two types of arguments stems from the link that the author or expositor of the argument establishes between the premises and the conclusion.

The argument is deductive if the author thinks that the truth of the premises absolutely determines the validity of the conclusion (owing to the definition, logical entailment, logical structure, or mathematical necessity).

If the author of the argument does not believe that the truth of the premises absolutely determines the truth of the conclusion, but thinks that the truth of the premises offers strong cause to consider the conclusion true, then the argument is inductive.

An inductive argument is one in which the arguer's only goal is to establish or raise the likelihood of its conclusion. The premises of an inductive are only meant to be strong enough that if they are true, the conclusion is unlikely to be wrong. A successful inductive argument does not have a standard word. However, unlike deductive reasoning, its success or power is a question of degree. A deductive argument is either valid or invalid.

2.2 Trigonometry

Trigonometry is the study of the relationship between the sides and angles of a triangle. The word "trigonometry" comes from the Greek words trigono (τρ΄ιγωνο), which means "triangle", and metro (μϱτρ΄ω), which means "measurement". Ancient Greeks like Hipparchus and Ptolemy used trigonometry to study astronomy around 150 BC-200 AD, its history is much older. For example, around 1650 BC, Egyptian scribe Ahmes drew a trigonometric function, Michael Corral (2009). Trigonometric functions (or circular functions) are the basic mathematical objects that all high school and undergraduate students studying science try to tame. Referencing them usually reminds me of one of the things that makes it difficult to remember the connections that connect them, the rules of calculation, and so on. This is done through a hexagonal figure with vertices marked by the ratio of six basic trigonometric functions. This practice was taught in a high school in China a few years ago.

2.3 History of Trigonometry

Triangles were first studied in Egyptian mathematics (Rhind Mathematical Papyrus) and Babylonian mathematics in the second millennium BC. Trigonometry was also used extensively in Kushite mathematics. The systematic study of trigonometric functions originated in Hellenistic mathematics and eventually made its way to India as part of Hellenistic astronomy. The study of trigonometric functions thrived in Indian astronomy during the Gupta era, thanks in large part to Aryabhata (sixth century CE), who developed the sine function. Trigonometry was studied in Islamic mathematics throughout the Middle Ages by mathematicians such as Al-Khwarizmi and Abu al-Wafa. In the Islamic world, where all six trigonometric functions were recognized, it formed a separate discipline. Beginning with Regiomontanus in the Renaissance, translations of Arabic and Greek writings led to trigonometry being recognized as a discipline in the Latin West. Modern trigonometry evolved during the Western Age of Enlightenment, beginning with 17th-century mathematics (Isaac Newton and James Stirling) and culminating with Leonhard Euler (1748).

2.3 Trigonometry Identities:

There are several trigonometric identifiers, that is; Equation valid for any angle equal to 2, used in trigonometric studies So far we have known some relationships between trigonometric functions. For example, we know mutual relations. Such equations are called homogeneous, and in this section we will discuss some trigonometric congruence, that is, homogeneity with respect to trigonometric functions. These identifiers are often used to simplify complex expressions or equations. For example, one of the most useful trigonometric identifiers is:

The sine of a given angle "$\boldsymbol{\theta}$" in a right-angled triangle is the ratio of its Opposite and Hypotenuse

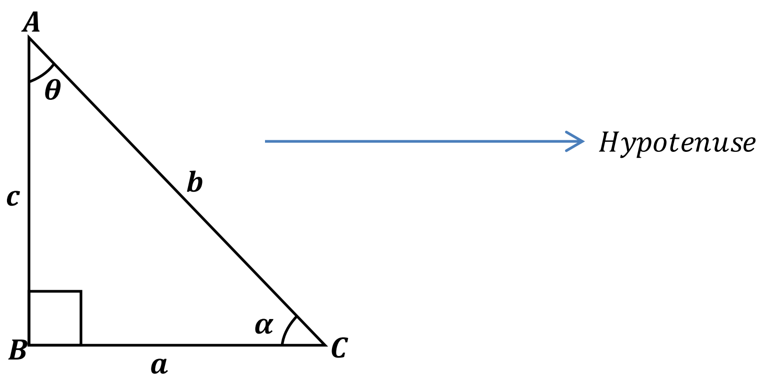


Fig1. Right angle triangle

Hypotenuse is the longest side (edge) of a right-angled triangle while the Opposite is depending on the given angle. Example; if the given angle is "$\boldsymbol{\theta}$" then side "$\boldsymbol{a}$" is said to be the Opposite while side "$\boldsymbol{c}$" is said to be the Adjacent.

Mathematically: $$\sin\theta = \frac{Opposite}{Hypotenuse} = \frac{a}{b} \quad (1)$$

1- The cosine of a given angle "$\boldsymbol{\theta}$" in a right-angled triangle is the ratio of its Adjacent and Hypotenuse.

$$\cos\theta = \frac{Adjacent}{Hypotenuse} = \frac{c}{b} \quad (2)$$

2- The tangent of a given angle "$\boldsymbol{\theta}$" in a right-angled triangle is the ratio of its Opposite and Adjacent.

$$\tan\theta = \frac{Opposite}{Adjacent} = \frac{a}{c} \quad (3)$$

Reciprocal of Trigonometric Ratios

The reciprocal of the sine of a given angle "$\boldsymbol{\theta}$" in a right-angled triangle is called $\boldsymbol{cosecant\ \theta}$ also known as $\boldsymbol{cosec\ \theta}$. In other words $\boldsymbol{cosecant\ \theta}$ is the multiplicative inverse of $\mathbf{sin}\,\boldsymbol{\theta}$

Mathematically;

$$\boldsymbol{cosec\ \theta} = \frac{\mathbf{1}}{\mathbf{sin}\,\boldsymbol{\theta}} \qquad (\mathbf{4})$$

The reciprocal of the cosine of a given angle "$\boldsymbol{\theta}$" in a right-angled triangle is called $\boldsymbol{secant\ \theta}$ also known as $\boldsymbol{sec\ \theta}$. In other words $\boldsymbol{secant\ \theta}$ is the multiplicative inverse of $\mathbf{cos}\,\boldsymbol{\theta}$

$$\boldsymbol{sec\ \theta} = \frac{\mathbf{1}}{\mathbf{cos}\,\boldsymbol{\theta}} \qquad (\mathbf{5})$$

The reciprocal of the tangent of a given angle "$\boldsymbol{\theta}$" in a right-angled triangle is called $\boldsymbol{gent\ \theta}$. Also known as $\boldsymbol{cot\ \theta}$. In other words $\boldsymbol{cotangent\ \theta}$ is the multiplicative inverse of $\mathbf{tan}\,\boldsymbol{\theta}$.

$$\boldsymbol{cot\ \theta} = \frac{\mathbf{1}}{\mathbf{tan}\ \boldsymbol{\theta}} \qquad (\mathbf{6})$$

2.0 Calculus

Calculus is simply the mathematics of movement and change. Calculus is the correct math to apply when there is movement or growth and there are various forces in the work that produces acceleration. Analysis was first invented to meet the mathematical needs of scientists in the 16th and 17th centuries, primarily the needs of mechanical properties. Analysis provides a solid foundation for learning differential equations. Basically, the analysis has two components: differential integral and integral calculus.

2.1 Differential calculus

Differential calculus deals with the problem of calculating the rate of change. This allowed humans to define the slope of the curve, calculate the velocity and acceleration of the moving object, find the firing angle that gives the cannon the maximum range, and predict when the planet will be closest or farthest away.

2.2 Integral calculus

Integral calculus deal with the problem of determining a function from information about its rate of change. This allows people to calculate the future position of an object from their current position and the knowledge of the forces acting on it, find areas of irregular regions in the plane, measure the length of the curve, and any solid.

Today, the expansion of calculus and its mathematical analysis is certainly widespread, and the mathematicians, physicists, and astronomers who first invented this subject are, as you are, the many that it solves. You will certainly be surprised and pleased to see the problem. Solving Today, there are many areas used in mathematical models that understand the universe and the world around us. To find the derivative of the basic trigonometric function, some trigonometric identities need to work.

Example:

The derivative of the function $y = f(x)$ is $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$ From the trigonometric identities;

(i)-$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$,

(ii)- $\cos A - \sin B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

The limit of the function $\frac{\sin\theta}{\theta}$, as θ (measured in radians) approaches zero , the function $\frac{\sin\theta}{\theta}$ tends to 1. Mathematically written as $\lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1$

The above result can be justified by choosing values of θ closer and closer to zero and examining the behavior of $\frac{\sin\theta}{\theta}$.

$\boldsymbol{\theta}$	$\boldsymbol{\sin\theta}$	$\boldsymbol{\frac{\sin\theta}{\theta}}$
1	0.84147	0.84147
0.1	0.09983	0.99833
0.01	0.00999	0.99983

Table 1: Shows the value of $\frac{\sin\theta}{\theta}$ as θ tends to zero is 1

The above results can be verified with calculator to appreciate that the value of $\frac{\sin\theta}{\theta}$ approaches 1 as θ tends to zero.

These results are use in order to find the derivatives of $f(x) = \sin x$ from the first principles.

(a) Given that $f(x) = \sin x$, to find the derivative

Let $f(x) = \sin x$ ………………………….. 1 and

$f(x+\delta x) = \sin(x+\delta x)$……………… 2

Subtracting equation 1 from 2

$$f(x+\delta x) - f(x) = \sin(x+\delta x) - \sin x$$

The right hand side is the difference of two sine terms. Hence,

the trigonometric identity $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$, will be used.

$$\sin(x+\delta x)-\sin x = 2\cos\frac{(x+\delta x)+x}{2}\sin\frac{(x+\delta x)-x}{2}$$

$$=2\cos\frac{2x+\delta x}{2}\sin\frac{\delta x}{2}$$

$$=2\cos\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$

Then, using the definition of the derivative

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\frac{dy}{dx}=\frac{2\cos\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\delta x}$$

$$=\cos\left(x+\frac{\delta x}{2}\right)\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

Now let δx tends to zero. Consider the term $\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$ and use the result that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$with $\theta=\frac{\delta x}{2}$.

Which implies that $\lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}=1$

$$=\cos\left(x+\frac{\delta x}{2}\right)$$

Further, $\lim_{\delta x\to 0}\cos\left(x+\frac{\delta x}{2}\right)$

$$\frac{dy}{dx}=\cos x$$

(b) The derivative of $f(x)=\cos x$.

Here, let $f(x)=\cos x$ ……………. (1)

So, $f(x+\delta x)=\cos(x+\delta x)$……..(2)

Subtracting equation (1) from (2)

$$f(x+\delta x)-f(x)=\cos(x+\delta x)-\cos x$$

The right hand side is the difference of two cosine terms. Hence, we use the trigonometric identity;

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

To write this in an alternative form.

$$\cos(x+\delta x) - \cos x = -2\sin\frac{x+\delta x+x}{2}\ sin\frac{\delta x}{2}$$

$$= -2\sin\frac{2x+\delta x}{2}\sin\frac{\delta x}{2}$$

$$= -2\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$

Then, using the definition of the derivative;

$$\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$= \frac{-2\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\delta x}$$

The factor of 2 can be moved as before, in order to write this in an alternative form:

$$\boldsymbol{\frac{dy}{dx}} = -\frac{\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\boldsymbol{\delta x/2}}$$

$$\boldsymbol{= -sin\left(x+\frac{\delta x}{2}\right)\frac{sin\,\frac{\delta x}{2}}{\frac{\delta x}{2}}}$$

Similarly, let δx tend to zero;

$$\lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}} = 1$$

Further,

$$\lim_{\delta x\to 0} -\sin\left(x+\frac{\delta x}{2}\right) = -\sin x$$

Thus, $\frac{dy}{dx} = -\sin x$

From the above differentiation using the first principle, the derivatives of the functions $f(x) = \sin x$ and $f(x) = \cos x$ respectively.

3.0 RESULT

3.1 The hexagon of the six trigonometric functions

The six trigonometric functions introduced above are placed at the corners of the hexagon, and the very good relationships between them can now be "confirmed" and memorized in the image. List them and comment. For perfection, add a 1 in the center of the hexagon.

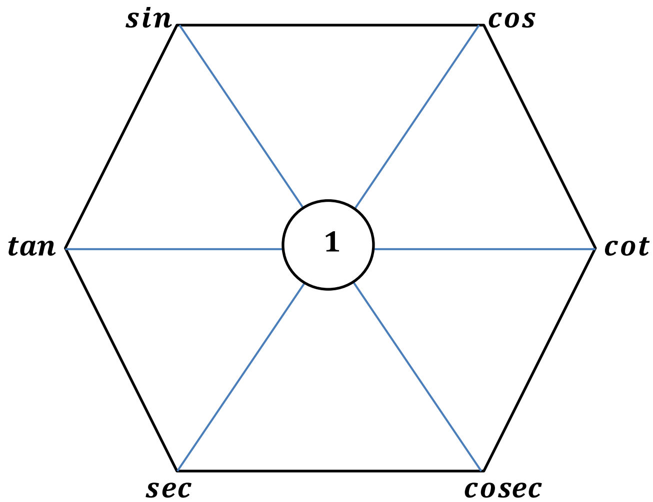


Fig. 2 Hexagon of the six trigonometric functions

The identities can be read from the super-hexagon diagram as follows:

Consecutive functions: The function placed at the apex is the quotient of the following two functions (move right or left). For example, a tangent is a sine and a cosecant (or a secant to a cosecant) quotient, and a secant is a secant divided by a tangent and a cosecant. Is the quotient of the sine divided by the secant. There are 12 relationships of this kind.

If there are three consecutive functions around a hexagon in either direction, the first function is equal to the second function divided by the third function.

3.2 Clockwise Direction

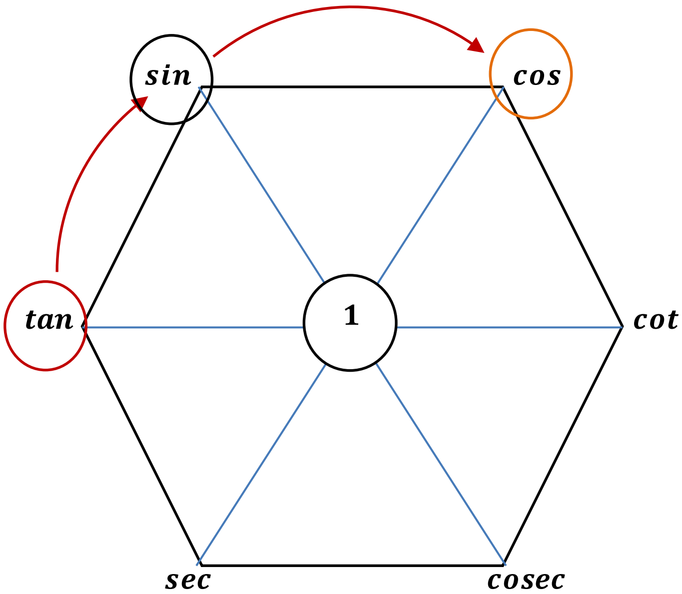


Fig. 3 Hexagon of the six trigonometric functions from Tan moving clockwise direction

(i) - $\tan x = \frac{\sin x}{\cos x}$

(ii) - $\sin x = \frac{\cos x}{\cot x}$

(iii)- $\cos x = \frac{\cot x}{\operatorname{cosec} x}$

(iv) - $\cot x = \frac{\operatorname{cosec} x}{\sec x}$

(v) - $\operatorname{cosec} x = \frac{\sec x}{\tan x}$

(vi)- $\sec x = \frac{\tan x}{\sin x}$

3.3 Anti-clockwise Direction

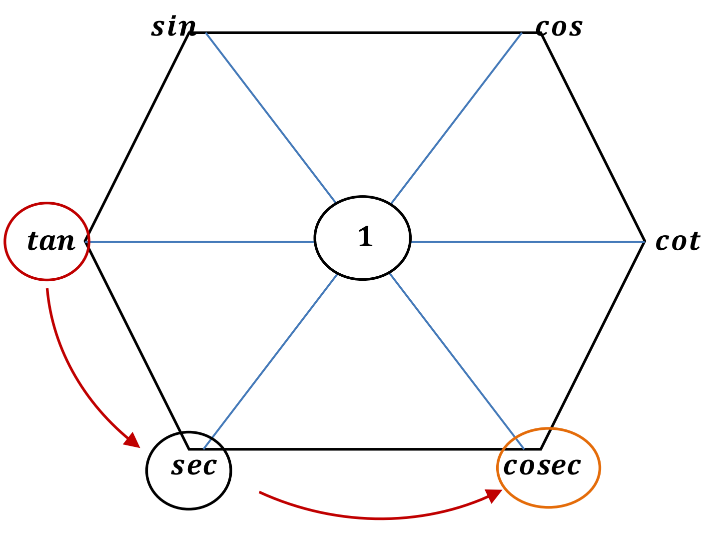


Fig. 4 Hexagon of the six trigonometric functions from Tan Moving anticlockwise direction

(vii)- $\tan x = \frac{\sec x}{\operatorname{cosec} x}$

(viii)- $\sec x = \frac{\operatorname{cosec} x}{\cot x}$

(ix) - $\boldsymbol{cosec\ x = \frac{\cot x}{\cos x}}$

(x)- $\boldsymbol{\cot x = \frac{\cos x}{\sin x}}$

(xi)- $\boldsymbol{\cos x = \frac{\sin x}{\tan x}}$

(xii)- $\boldsymbol{\sin x = \frac{\tan x}{\sec x}}$

3.4 Diametrically opposite functions

The function placed at the vertex is the inverse of the opposite function. That is, the product of the opposite functions is equal to the median 1. There are three such relationships. The functions at both ends of the diagonal are reciprocals. That is, if you multiply the two functions at the ends of the diagonal, you get a central "1".

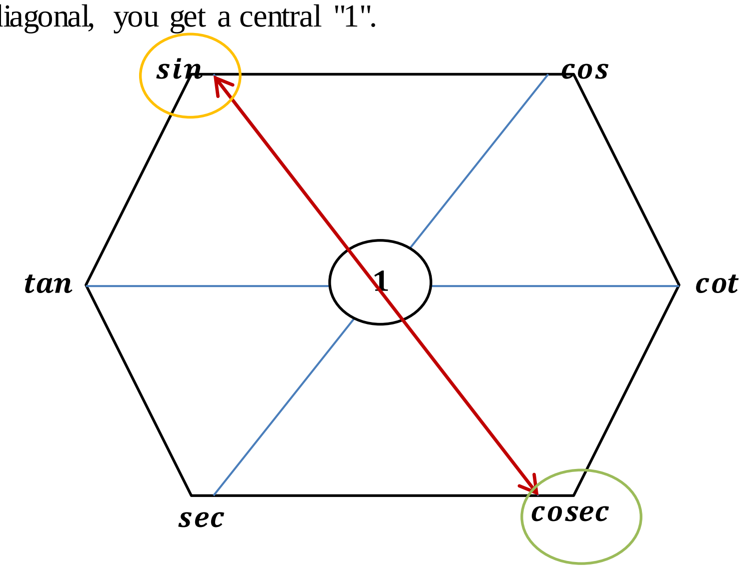


Fig. 5 Hexagon of the six trigonometric functions, Considering opposite functions

(xiv)- $\boldsymbol{cosec\ x = \frac{1}{\sin x}}$

(xv)- $\boldsymbol{\sec x = \frac{1}{\cos x}}$

(xvi)- $\boldsymbol{\cot x = \frac{1}{\tan x}}$

These also imply that;

(xvii)- $\boldsymbol{\sin x\ cosec\ x = 1}$

(xviii)- $\boldsymbol{\cos x\ \sec x = 1}$

(xix)- $\boldsymbol{\tan x\ \cot x = 1}$

3.5 Neighboring functions

The function placed at the vertex is the product of two adjacent functions (one from the right and the other from the left). For example, cosine is the product of sine and cotangent, cotangent is the product of cotangent and secant. There are six relationships of this type. Each function is equal to the product of adjacent functions.

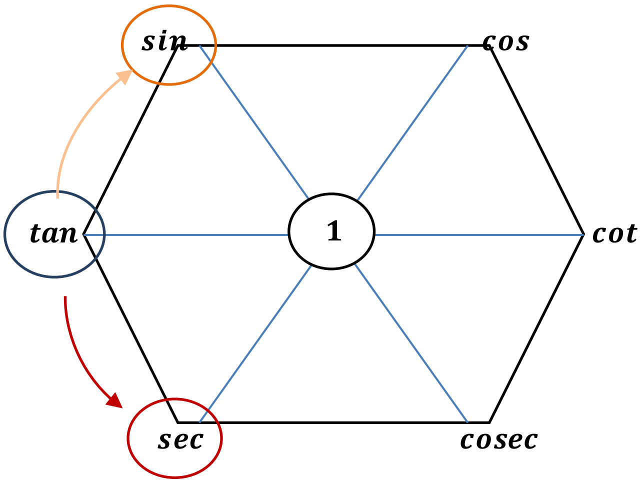


Fig 6. Hexagon of the six trigonometric functions considering neighboring functions

(xx)- $\tan x = \sin x \sec x$

(xxi)- $\sin x = \cos x \tan x$

(xxii)- $\cos x = \sin x \cot x$

(xxiii)- $\cot x = cosec\ x \cos x$

(xxiv)- $cosec\ x = \cot x \sec x$

(xxv)- $\sec x = cosec\ x \tan x$

3.6 The product of any three alternating function is "1".

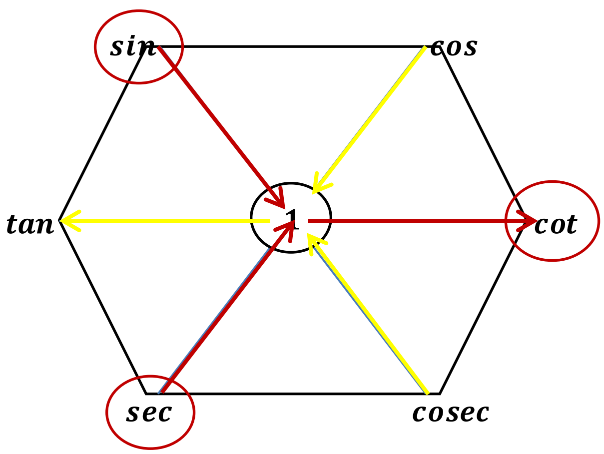


Fig. 7 Hexagon of the six trigonometric functions showing the Product of any three alternating function as "1"

(xxvi)- $\cos x\ \ cosec\ x\ \tan x = 1$

(xxvii)- $\sin x \sec x \cot x = 1$

3.7 Addition of function squares: The hexagons of the six trigonometric functions have three equilateral triangles oriented north, southeast, and southwest. The functions located at the two outer corners of these three triangles and the 1 (constant function) located at the center corner follow the following rules according to the arrows. The sum of the squares of the first two corners is equal to the square of the third corner.

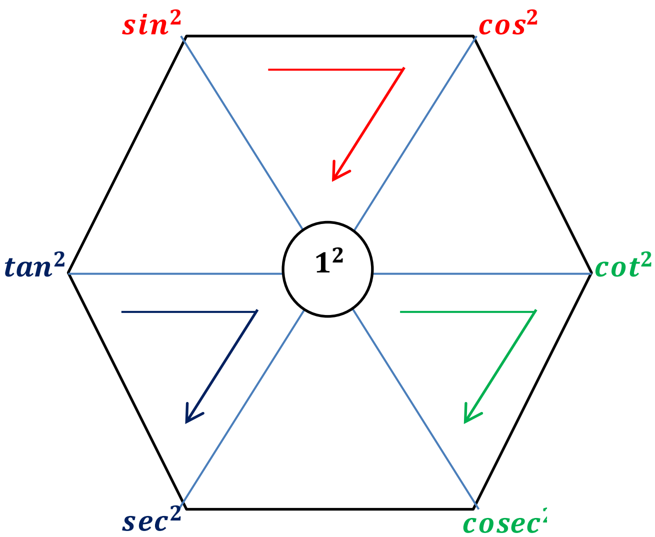


Fig. 8 Hexagon of the sum of squares of the first two vertices equals the square of the third vertex

(xxviii)- $\boldsymbol{sin^2 x + cos^2 x = 1}$ **(xxx)-** $\boldsymbol{tan^2 x + 1 = sec^2 x}$

(xxix)- $\boldsymbol{1 + cot^2 x = cosec^2 x}$

Note:

i- Those three empty triangles in the hexagon. The Pythagorean identities may be easily recalled using them. The square of the function at the apex of the triangle is equal to the sum of the squares of the two functions at the base of the triangle.
ii- $1^2 = 1$

3.8 The co-function identities can be reconstructed from the rows within the hexagon.

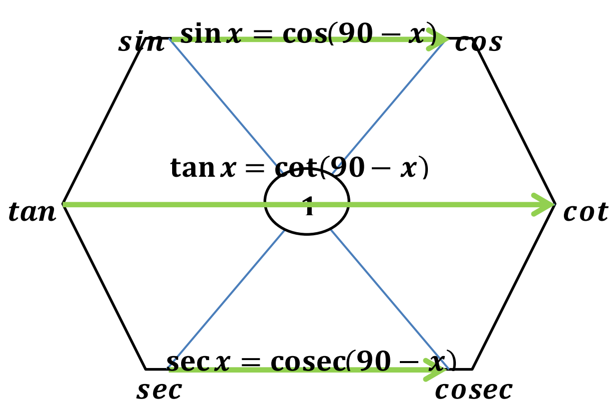


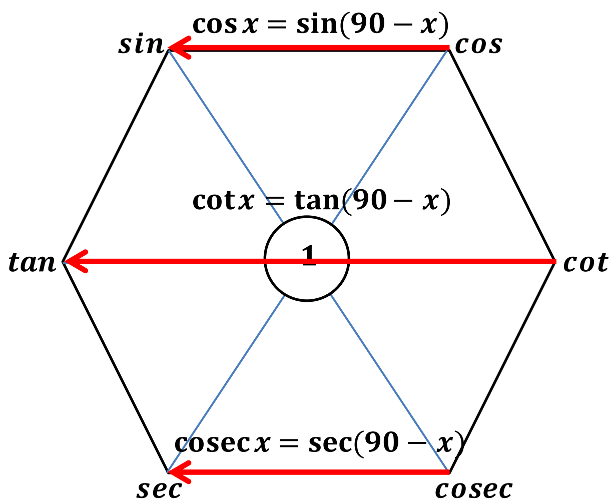
Fig. 9 Reconstruction of identities from the rows within the hexagon

(xxxi)- $\sec x = cosec(90 - x)$

(xxxii)- $\tan x = \cot(90 - x)$

(xxxiii)- $\sin x = \cos(90 - x)$

Fig.10 Reconstruction of identities from the rows in reverse order within the hexagon.



(xxxiv)-$\cos x = \sin(90 - x)$

(xxxv)- $\cot x = \tan(90 - x)$

(xxxvi)- $cosec\ x = \sec(90 - x)$

1.0 SUPER HEXAGONIN DIFFERENTIAL CALCULUS

When entering the differential calculus,, the hexagon also helps to remember the derivative of trigonometric functions such as 〖y = sin〗 x. Draw a vertical line through the center of the hexagon. The function on the left side of the hexagon has a positive derivative and the function on the right side has a negative derivative.

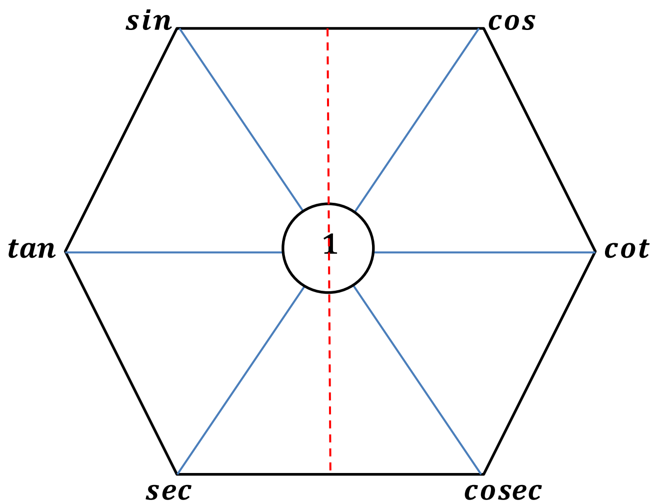


Fig. 10 Showing Hexagon used in deducing the differential calculus and the vertical red dotted line bisects the hexagon into two parts, right hand side and the left hand side. The derivative of the trigonometric function on the left hand side are all positive while the derivative of the trigonometric functions on the right hand side are all negative.

The derivatives at the top row (row 1) are easy: They point to each other.

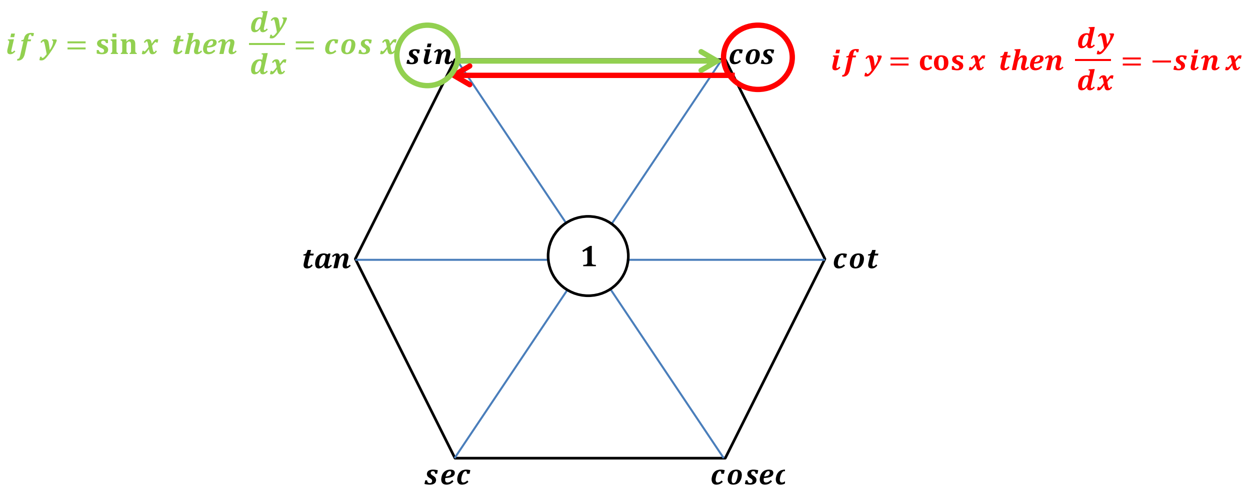
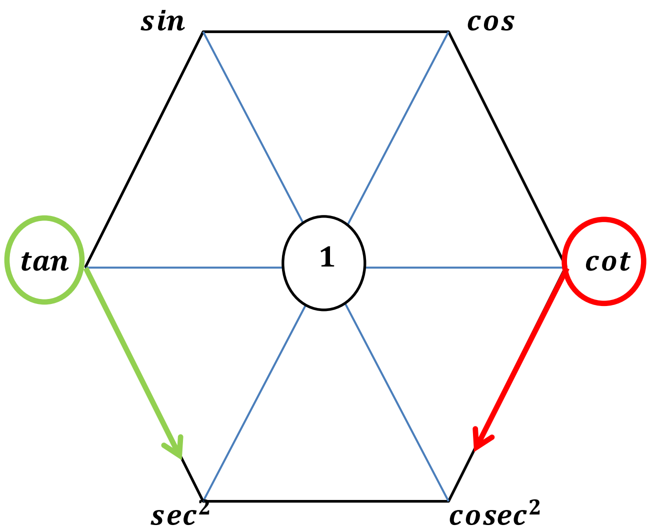


Fig. 11 Showing Hexagon used in deducing the differential calculus of $\sin x$ and $\cos x$

i- If $y = \sin x$ then $\frac{dy}{dx} = \cos x$ ii- If $y = \cos x$ then $\frac{dy}{dx} = -\sin x$

For the derivative of the middle row functions, take the function below and square it.



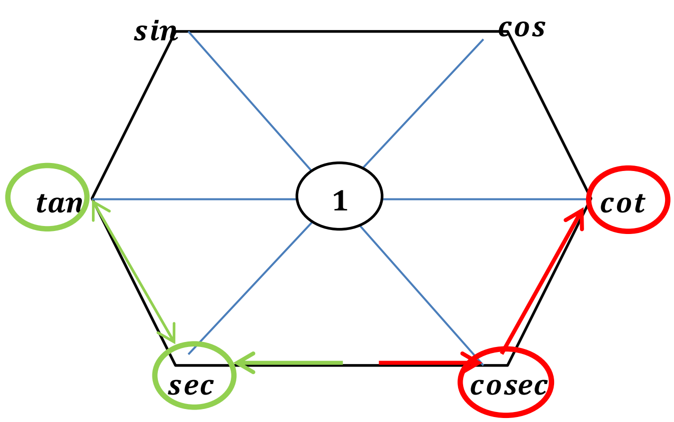
$if\ y = \tan x\ then\ \frac{dy}{dx} = sec^2\,x$ $\quad$ $if\ y = \cot x\ then\ \frac{dy}{dx} = -cosec^2 x$

Fig 12. Showing Hexagon used in deducing the differential calculus of $\tan x$and $\cot x$

i- If $\boldsymbol{y = \tan x}$ then $\frac{dy}{dx} = \boldsymbol{sec^2 x}$

ii- If$\boldsymbol{y = \cot x}$ then$\frac{dy}{dx} = \boldsymbol{-cosec^2}$

For the derivative of the bottom row functions, take the original function and the function above it.



$if\ y = \sec x\ then\ \frac{dy}{dx} = sec\,x\,tan\,x$ $\quad$ $if\ y = \operatorname{cosec} x\ then\ \frac{dy}{dx} = -cosec\,x\,cot\,x$

Fig. 13Showing the Hexagon used in deducing the differential calculus of $\sec x$ and $cosec\ x$

iii- If $\boldsymbol{y = \sec x}$ then $\frac{dy}{dx} = \boldsymbol{\sec x \tan x}$

iv- If $\boldsymbol{y = cosec\ x}$ then $\frac{dy}{dx} = \boldsymbol{-cosec\ x \cot x}$

5.0 Conclusion

The application of super hexagon in the trigonometric identities helps to facilitate the processes of finding the derivatives of trigonometric function. It also helps the learner to visualize the learned content which will in-turn help the learner to easily memorize the trigonometric identities. Hence, it is recommended that Nigerian Mathematicians and Mathematics educators should integrate the super Hexagon method in the teaching of trigonometry at all levels of learning.

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